Perturbation-Asymptotic Series Approach for an Electromagnetic Wave Problem in an Epsilon Near Zero (ENZ) Material

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ABSTRACT Electromagnetic waves present very interesting features while the permittivity of the environment approaches to zero. This property known as ENZ (Epsilon Near Zero) has been analysed with the perturbation approach-asymptotic analysis method. Wave equations have been solved by space transformation instead of phasor domain solution and the results compared. Wave equation is non-dimensionalised in order to allow asymptotic series extension. Singular perturbation theory applied to the Wave Equation and second order series extension of electromagnetic waves have been done. Validity range of the perturbation method has been investigated by modifying parameters.

INDEX TERMS perturbation; asymptotic expansion, Epsilon Near Zero (ENZ), non dimensionalization, dominant balancing, space transformation, electromagnetic wave equation.

I. INTRODUCTION

THIS study aims to introduce an alternative approach to the solution of Electromagnetic wave equation, which is asymptotic series expansion-perturbation method. Perturbation method has been widely used in many areas in a spectrum from nuclear physics to astronomy. A wide amount of works explain basic and advanced explanations with various applications of this method. [1]–[3].

Perturbation, derived from the word "perturb", means to change some parameters of a process slightly. "Small or slight change" is a major factor in perturbation [4]. On the other hand, Epsilon Near Zero (ENZ) is an important and innovating concept in electromagnetics, which is postulated by Lorentz [5], [6]. In this work, slight deviations of relative permittivity within the range "0" to "1", inspired for ENZ (Epsilon Near Zero) concept. Thus, as a novel approach, an Electromagnetic transmission problem inside an ENZ material has been solved with perturbation theory by assigning perturbation parameter as the permittivity of an ENZ material.

Within the study, basic properties of the perturbation theory is given with an emphasis on singularity. Singular perturbation is an interesting feature of this theory since slight deviations causes big changes on the solution. Thus, going one step further, singular perturbation theory is applied on the damping electromagnetic wave, that travels inside an ENZ material. But there was an important challenge: additive terms in the asymptotic series extension has different powers. Only way to sum them up is to get rid of their units. This is achieved by making parameters unitless [7] .

Relative permittivity, ε_r was chosen to be a perturbation parameter. To introduce singularity, this parameter has to be a "main actor" of the equation; which means, it has to be a coefficient of the highest order derivative term. In this way perturbation parameter (relative permittivity), implements a singularity while it approaches to zero and its existence becomes critical for the description of solution pattern. In order to have this form of the equation and assign perturbation parameter, ε_r as a coefficient to the largest differential term, space (β) transformation has been applied to the wave equation instead of frequency transformation. Next step was to be sure that this space transformation technique solution [6], which drops the space dependency from $(x, y, z; t)$ to (t), has the similar solution set with the classical way of solution in phaser domain (frequency) transformation.

In the second part of the text, space transformed equation solved with perturbation concepts. As explained above, since perturbation theory basically depends on the asymptotic series expansion of the equation, the parameters are needed to be deunitized or non-dimensionalised [7]. In this stage,since de-unitization parameters are not unique, they are chosen in a way to allow application of perturbation solution for a singular, second order differential equation. In the study, two different de-unitization parameter set used and demonstrated

TABLE 1. Typical values for the verification of space and freq. transform equivalence

Electric Field (E. Field) Desc. Unit	Notation
Vector Electric Field (V/m)	$\mathbf{E}$
"x" component of E. Field (V/m)	E_x
Freq. Fourier Transf. of E_x Field	E_ω
Space Fourier Transf. of E_x Field	E_β
Unitless E. Field $(unitless)$	$\tilde{E}$
Perturb. Sol'n, Inner E. Field $(unitless)$	E^i
Perturb. Sol'n, Outer E. Field $(unitless)$	E^o
Perturb. Sol'n, Total E. Field $(unitless)$	$\hat{E}$

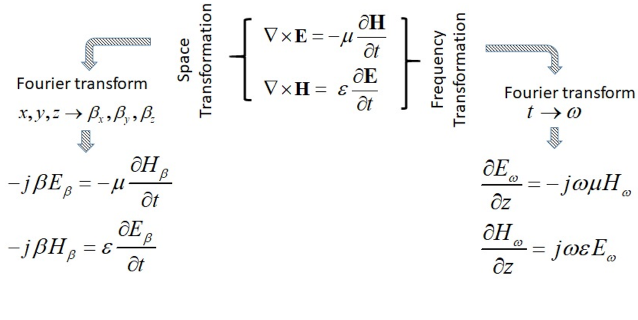

FIGURE 1. Space and Frequency Fourier Transformation of an EM wave

to match with the original equation.

So far, two transformations are mentioned for Electric Field wave, damped propagation function: "Space (β)" and "non-dimensionalization" transformations. These transformations are shown to comply with results of traditional phasor domain solution method. After obtaining a "good form" equation which can be solved by perturbation methods and asymptotic series expansion concept, another transformation needed to overcome the inherited singularity. This is implemented by Dominant Balance Method, which expands the solution interval where the perturbation parameter is effective and transformation parameters are determined.

In order to have a better readability, Electric Field variable, subjected to transformations and notations after these transformations that are used throughout the study, summarized in the Table 1.

Concepts mentioned above introduced more in detail with illustrations of simulation results within the text.

A. EPSILON NEAR ZERO (ENZ) MATERIALS

Recently, epsilon-near-zero (ENZ) materials have been proposed as a way to achieve efficient sub wavelength electromagnetic energy transport. [8], [9] ENZ materials have near-zero dielectric constant (epsilon) in the effective medium limit resulting in an unusually large effective wavelength even at optical frequencies is thus expected to enable energy squeezing and transfer through sub wavelength channels with lower losses over larger bandwidth. The optical properties of materials which is the response to an applied external electromagnetic field can be expressed using the relative permittivity(ε_r) and the relative permeability (μ_r) of the medium [10], [11].

B. SPACE TRANSFORMATION

Fourier Transformation on frequency is frequently used to simplify an equation with frequency dependency [12]. In this study, space transformation will be used as to be explained in the related section. The function for the solution is chosen to be; $f(t,\beta) = A\cos(\beta z - \omega t)$.

Following Fourier transformations applied to $f(t;\beta)$:

i) Frequency Transformation (Phaser Domain):

$$F(\omega) \equiv F_w = \int_{-\infty}^{\infty} f(t,z)\, e^{-j\omega t} dt \tag{1}$$

ii) Space Transformation (β-Domain):

$$F(\beta) \equiv F_\beta = \int_{-\infty}^{\infty} f(t,z)\, e^{-j\beta z} dz \tag{2}$$

For space transformation 1^{st} and 2^{nd} derivative transformations defined as in conventional case:

$$F_\beta\{\frac{\partial E_x}{\partial t}\} = (j\beta)E_\beta, \\ F_\beta\{\frac{\partial^2 E_x}{\partial t^2}\} = (j\beta)^2 E_\beta = -\beta^2 E_\beta \tag{3}$$

where E_β is the space Fourier transformation (2) of the Electric field, E_x. In this study, space transformation is used and solutions validity is checked with the frequency transformation.

A schematic flow for frequency and space transformation is given in Fig. 1 where β subscript for Electric (E_x) and Magnetic (H_y) fields represents space Fourier Transformation, and ω subscript stands for frequency Fourier Transformation.

C. PERTURBATION THEORY

Perturbation theory is a collection of mathematical methods used to obtain approximate solution to "hard" problems that do not have closed-form analytical solutions. These methods reduces a hard problem to an infinite sequence of relatively easy problems that can be solved analytically. Perturbation problems depend on a small positive parameter.

This parameter, when applied to the highest order term in the equation, affects the problem in such a way that the solution varies rapidly in some region of the problem domain and slowly in other parts. [2], [4]. The region where the solution varies rapidly is called the inner region. A variety of solution techniques are used to extent this region and to provide a solution for this very narrow but very fast changing and important region. Beyond the inner region, whose boundaries are also determined with literally well known techniques, the singularly perturbed boundary value problem possesses boundary or interior layer. regions of rapid change in the solution near the end points or some interior points with interval as detailed in [2], [13].

A question may arise, why to use perturbation techniques while there are computers that can solve very complicated non-linear, inhomogeneous, multidimensional problems. What is the need for such a method? The answer is that the insight capability in to the physics of the problem. The principal objective when using perturbation methods, is to provide a reasonably accurate expression for the solution, with an understanding of the physics of a problem. Also, results can be used in conjunction with the original problem, to obtain more efficient numerical procedures for computing the solution [3], [14]. Boundary value problem which has a perturbation leading to singularity, possesses boundary or interior layer i.e. regions of rapid change in the solution near the end points or some interior points with width $O(1)$ as $\varepsilon \to 0$.

II. DAMPED WAVE EQUATION SOLUTION

A. GENERAL APPROACH

One dimensional wave equation can be defined in general as follows [15], [16]:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial z^2} \tag{4}$$

Solution to (4) is defined by D'Alambert as:

$$u(z,t) = f(z - vt) + f(z + vt) \tag{5}$$

where f is a differentiable, arbitrary function with an argument of $(z - vt), (z + vt)$; causing the function to "travel" to left and right. If the function is chosen as a sinusoid which travels to "right" and solved for different time values, the argument of the function do not change wrt an imaginative observer 5.
Since the argument of the function is constant, its derivative wrt time is zero and "v" is the propagation speed of the wave, which is given by $1/\sqrt{\mu\varepsilon}$.
Selecting function as a sinusoid, its argument turns out:

$$(z - vt) \Leftrightarrow (\beta(z - vt)) = (\beta z - \omega t) \tag{6}$$

solution function as; $f(z,t) = A\cos(\beta z - \omega t)$. where, for lossless systems, $\beta = k = 2\pi/\lambda$ is the wave number, λ is the wavelength and ω is the angular frequency, $\omega = \beta v = 2\pi f$.
Since instead of time, space transformation is used, space derivative wrt. z component is applied to the argument:

$$\begin{gathered} \frac{\partial}{\partial z}(\beta z - \omega t) = 0, \quad \beta - \omega \frac{\partial t}{\partial z} = 0 \\ \frac{\partial t}{\partial z} = \frac{1}{v}, \quad \beta - \omega(\frac{1}{v}) = 0, \quad \beta = \frac{\omega}{v} \end{gathered} \tag{7}$$

which gives the same result above. Wave equation for a Lossy Electric field vector, derived from Maxwell Equations in a lossy media:

$$\begin{gathered} \nabla^2 \mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \\ \mathbf{E} = E_x \hat{a}(x) + E_y \hat{a}(y) + E_z \hat{a}(z)) \end{gathered} \tag{8}$$

Its simplified version for a wave travelling in z direction, with x component only:

$$\begin{gathered} \frac{\partial^2 E_x}{\partial z^2} - \mu\sigma \frac{\partial E_x}{\partial t} - \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} = 0\,, \\ E_x(0) = 0; E_x'(0) = C/\varepsilon_r \end{gathered} \tag{9}$$

Initial condition for the first derivative in (9) is the displacement current. (9) is the function that is to be solved by using space transformation and perturbation techniques. A generic solution to this equation is given in Fig. 2. It should be noted that the wave function itself is dispersive, due to frequency dependency of lossy characteristic of the environment ($\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$).
For a lossless, non magnetic medium ($\mu_r = 1$), phase velocity is independent of frequency. However, In dispersive (lossy) environment, as $\partial\omega \to 0$, instead of phase velocity, group velocity defined. Relation between the two coordinates; "distance z(m)" and "time t(s)" is;

$$\begin{gathered} z = vt = \frac{\omega}{\beta} t \\ similarly,\ t = \frac{z}{v} = z\frac{\beta}{\omega}, \quad \frac{\partial E_x}{\partial t} = \frac{\beta}{\omega}\frac{\partial E_x}{\partial z} \\ phase\ velocity: v_p = \frac{w}{\beta}\,(m/s) = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} \end{gathered} \tag{10}$$

Group velocity may be higher (anomalous dispersion) or lower (normal dispersion) than phase velocity. These facts causes some deviations from the equality conditions for frequency and space transformations, especially for the phaser terms. Below the Electric field equations will be solved by frequency and space transformation methods to show their similar characteristics, with small deviations defined above.

B. SOLUTION BY SPACE TRANSFORMATION

Equation (9) can be solved by using conventional frequency domain representation. However in this study, (9) will be solved by Fourier transformation in Space (x,y,z) domain, in order to obtain a function which is suitable to be solved by singular perturbation techniques [12]. If the transformations defined in (2), (3) are used for (9);

$$\begin{gathered} F_\beta\{\frac{\partial^2 E_x}{\partial z^2} - \mu\sigma \frac{\partial E_x}{\partial t} - \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}\} = 0\,, \\ = -\beta^2 E_\beta - \mu\sigma \frac{\partial E_\beta}{\partial t} - \mu\varepsilon \frac{\partial^2 E_\beta}{\partial t^2} = 0 \end{gathered} \tag{11}$$

where E_β is the space transformed version of E. Using (9);

$$\begin{gathered} \mu\varepsilon \frac{\partial^2 E_\beta}{\partial t^2} + \mu\sigma \frac{\partial E_\beta}{\partial t} + \omega^2 \mu\varepsilon E_\beta = 0 \\ \varepsilon_r \frac{\partial^2 E_\beta}{\partial t^2} + \frac{\sigma}{\varepsilon_0}\frac{\partial E_\beta}{\partial t} + \omega^2 \varepsilon_r E_\beta = 0 \end{gathered} \tag{12}$$

Where, $\varepsilon = \varepsilon_0\varepsilon_r$ is used and simple algebraic operations applied (12). Further simplified in representation as;

$$E'_\beta = \frac{\partial E_\beta}{\partial t}, \quad E''_\beta = \frac{\partial^2 E_\beta}{\partial t^2}$$
$$\varepsilon_r E''_\beta + \frac{\sigma}{\varepsilon_0}E'_\beta + \omega^2\varepsilon_r E_\beta = 0 \quad (13)$$
$$initial\ conditions;\ E_\beta(0) = 0; \frac{\partial E_\beta}{\partial t}(0) = C/\varepsilon_r$$

Equation (13) is an ODE and can be solved conventionally;

$$\text{characth. eqn}: \underbrace{\varepsilon_r}_{a} D^2 + \underbrace{\frac{\sigma}{\varepsilon_0}}_{b} D + \underbrace{\omega^2\varepsilon_r}_{c} D = 0$$
$$D_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \tilde{\alpha} \mp j\tilde{\beta} \quad (14)$$

Thus, solution for space transformation can be written as;

$$E_\beta(t) = A_\beta e^{-D_1 t} + Be^{D_2 t};$$
$$E_\beta(t) = A_\beta e^{\tilde{\alpha}t}e^{j\tilde{\beta}t} + Be^{-\tilde{\alpha}t}e^{-j\tilde{\beta}t} \quad (15)$$

Taking one solution which travels to right:

$$E_\beta(t) = A_\beta e^{-Dt};\ D = \tilde{\alpha} + j\tilde{\beta}$$
$$E_\beta = \left[e^{-\tilde{\alpha}t}e^{j\tilde{\beta}t}\right] \quad (16)$$
$$E_\beta(z,t) = A_\beta e^{-\tilde{\alpha}t}cos(\beta z - \tilde{\beta}t)$$

Constant, A_β can be found by using initial conditions given in (9).

C. SOLUTION BY FREQUENCY TRANSFORMATION

Frequency transformation (1),(2) was used to reduce and solve the same equation (9). Basically in this transformation, "$\frac{\partial^2 E_x}{\partial z^2}$" replaced by "$j\omega$" and "$\frac{\partial^2 E_\omega}{\partial z^2}$" replaced by "$-\omega^2$":

$$F_w\{\frac{\partial^2 E_x}{\partial z^2} - \mu\sigma\frac{\partial E_x}{\partial t} - \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}\} = 0\,,$$
$$\frac{\partial^2 E_\omega}{\partial z^2} - (j\omega\mu\sigma)E_\omega + (\omega^2\mu\varepsilon)E_\omega,$$
$$\frac{\partial^2 E_\omega}{\partial z^2} - (j\omega\mu\sigma - \omega^2\mu\varepsilon)E_\omega \quad (17)$$
$$let\ \gamma^2 = j\omega\mu\sigma - \omega^2\mu\varepsilon),\ \gamma = \sqrt{j\omega\mu\sigma - \omega^2\mu\varepsilon)},$$
$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$
$$\frac{\partial^2 E_\omega}{\partial z^2} - \gamma^2 E_\omega = 0$$

where E_ω, E_β are used to differentiate Frequency transformed and Space transformed versions of E Field. The parameter "γ represents complex wavenumber "k", for a lossy medium.

Relation between the Frequency and Space transformed versions of E Field are;

$$\text{E(z,t)} = Re\{e^{j\omega t}E_\omega\} = Re\{e^{j\beta z}E_\beta\} \quad (18)$$

using (17), conventional solution in frequency domain:

$$E_\omega(z) = Ae^{-\gamma z} + Be^{\gamma z}$$
$$\gamma = \alpha + j\beta \quad (19)$$
$$E_\omega(z) = Ae^{-\alpha z}e^{-j\beta z} + Be^{\alpha z}e^{j\beta z}$$

Taking the solution which travels to right yields;

$$E_\omega = A_w e^{-\alpha z}e^{-j\beta z}$$
$$E_\omega(z,t) = A_w e^{-\alpha z}cos(\beta z - wt) \quad (20)$$

Initial conditions, were given originally wrt to time (t). Thus, for frequency transformation solution, initial conditions has to be transformed to space domain while finding the constant A_w.

$$E_w(0) = 0;\ \frac{\partial E}{\partial t}(0) = \frac{C}{\varepsilon_r}$$
$$j\omega E_\omega(0) = C/\varepsilon_r, \quad E_\omega(0) = \frac{C}{j\omega\varepsilon_r} \quad (21)$$

Evaluating (20) and (21), constant found to be;
$A_\omega = C\beta/(\varepsilon_r\omega\gamma)$
Damping Electric field equation that propagates in the direction of "z" is given in Fig. 2

D. FREQUENCY TRANSFORMATION-SPACE TRANSFORMATION COMPARISON

Regarding the two transformed versions for the lossy wave equation, equations are obtained(18), (14). Below, these second order Ordinary Differential Equations (ODEs) are conventionally solved and plotted.
Constants throughout the calculations are; Permeability =(μ_0) $4\pi 10^{-7}$ H/m, Permittivity =(ε_0) $10^{-9}/(36\pi)$ F/m, Displacement Current constant, C = 10^9

Equations (16) and (20) shall refer to the same solution, Electric field. Below, these equations will be solved and plotted in order to show their similar behaviour. While frequency response solution basically depends on space and space transformation depends on time, they have to be on the same coordinate axis to be plotted together: Relation between time and space is;

$$t = \frac{z}{v},\ v = \frac{w}{\beta} \quad t = \frac{w}{\beta}z$$
$$For\ lossless\ case; v = 1/\sqrt{\mu\varepsilon}, \quad \beta = \omega\sqrt{\mu\varepsilon} \quad (22)$$

E. NONDIMENSIONALIZATION OF THE WAVE EQUATION

Due to the Principle of Dimensional Homogeneity (PDH), every additive term in an equation must have the same dimensions. Thus for Series expansions, in order to apply PDH, each item in the series shall be unitless since they do not have the same order. Similarly, since the perturbation theory introduces an (asymptotic) series expansion, removal of units from physical quantities by a suitable substitution of variables is needed. As a result, unitless terms (which can be added now) constitute a unitless equation, whatever any nonlinear operation (taking square or cube or higher order) applied to each term.
As a natural result of the process, non-dimensionalization has another advantage: to reduce number of variables (or natural parameters like $\varepsilon, \mu, \sigma, ..$) by combining them into a "super" variable. Similar case encountered here and a new

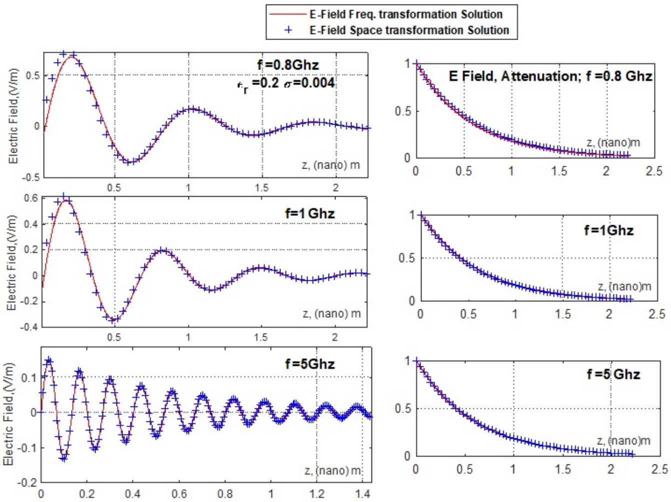

FIGURE 2. Electric field solution and attenuations by frequency and space domain approaches

TABLE 2. Units for the non-dimensionalization process of the wave equation

Parameter	Dimension (Unit)
Electric Field (E)	V/m
Time (t)	second (s)
Conductivity (σ)	s/(H.m)
Rel. permittivity (ε_r)	*unitless*
Permittivity (ε_0)	F/m
Permeability (μ_0)	H/m
Wave number (k)	1/m
Radial freq. ($\omega = 2\pi f$)	1/s
($\varepsilon\mu$)	s^2

perturbation parameter which is a combination of the other parameters of the system created. This has the advantages to analyse perturbation effect of all the parameters involved, as well. These concepts will be clarified in this section.
Wave equation form that will be used hereafter, which is space transformed, is repeated below for convenience:

$$\begin{gathered}\varepsilon_r E''_\beta + \frac{\sigma}{\varepsilon_0} E'_\beta + \omega^2 \varepsilon_0 E_\beta = 0 \\ with; E_\beta(0) = 0;\ E'_\beta(0) = C/(\varepsilon_r)\end{gathered} \tag{23}$$

The two variables (coordinates) of the system are E (V/m) and t (s). Other (natural) parameters are listed in Table II-E with their units.

In order to make the variables E and t unitless, values E_0, t_0 that has the same unit with original $E, (V/m)$ and $t, (s)$ has to be found. Since these values have same dimensions, the units cancel out and the resultant variable is unitless when their ratios are taken.
Representing unitless electric field with $\tilde{E}$, and unitless time variable as τ following definitions are valid:

$$\begin{gathered}\tilde{E} = \frac{E_\beta}{E_0}, \quad E_\beta = \tilde{E}E_0 \\ \tau = \frac{t}{t_0}, \quad t = \tau t_0\end{gathered} \tag{24}$$

Any selection for t_0 with unit "second" and any selection for E_0, with unit "V/m" are suitable for the non-dimensionalization process. Thus the solution for these constants are not unique. For example, $t_0 = 1/\omega$ is one of the solutions, since the result of this operation has a unit "s", (second), making time variable unitless when divided.
But for this study, without going into detail, after some trials following coefficients are found to be convenient for the rest of the work. All these coefficients below, are unitless.

$$\begin{gathered}t_{01} = \frac{\sigma}{\omega^2\varepsilon}\,[s], \quad \tau = \frac{\omega^2\varepsilon}{\sigma} t, \\ E_0 = \frac{\varepsilon_0 C}{\sigma}\ \texttt{[V/m]}, \quad \tilde{E} = (\frac{\sigma}{\varepsilon_0 C}) E_\beta, \\ alt.\ choice:\ t_{02} = \frac{\sigma}{\varepsilon}\,[s], \quad \tilde{\tau} = \frac{\sigma}{\varepsilon} t.\end{gathered} \tag{25}$$

Solutions were not unique; for time (t) variable, 2 different de-unitizing parameter, t_{01}, t_{02} with corresponding unitless time parametres $\tau, \tilde{\tau}$ are found and defined in (25).
It should be noted that for t_{01}, t_{02}, corresponding E_0's are found from initial condition equation:

$$\begin{gathered}\text{initial condition}: \varepsilon_r \frac{\partial E_\beta}{\partial t} = C \\ So,\ E_\beta = E_0 \tilde{E};\ t = t_{02}\tilde{t}, \quad \tilde{\tau}' = \frac{t}{t_{02}} = \frac{\sigma}{\varepsilon} t \\ \varepsilon_r \frac{E_0}{t_{02}} \frac{\partial \tilde{E}}{\partial \tilde{\tau}} = C, \quad E_0 = \frac{\varepsilon_0 C}{\sigma}\end{gathered} \tag{26}$$

As a crosscheck, unit of the E_0 can be verified by units given in Table (2); $(F/m)(m.s/V)(m.H/s) = (V/m)$; which is the desired result (note : [unit]$FH = [s^2]$).
First and second order derivatives can be calculated by using chain rule and values given in (26);

$$\begin{gathered}\frac{\partial E_\beta}{\partial t} = \frac{E_0}{t_0} \frac{\partial \tilde{E}}{\partial \tau} \\ \frac{\partial^2 E_\beta}{\partial t^2} = \frac{E_0}{{t_0}^2} \frac{\partial^2 \tilde{E}^2}{\partial \tau^2}\end{gathered} \tag{27}$$

Now, substituting these equivalences into (27),

$$\begin{gathered}\varepsilon_r \frac{E_0}{{t_0}^2} \tilde{E}'' + \frac{\sigma}{\varepsilon_0} \frac{E_0}{t_0} \tilde{E}' + \omega^2 \varepsilon_r E_0 \tilde{E} = 0 \\ (\frac{\omega\varepsilon}{\sigma})^2 \tilde{E}'' + \tilde{E}' + \tilde{E} = 0\end{gathered} \tag{28}$$

Defining $(\frac{\omega\varepsilon}{\sigma})^2$ as new perturbation parameter;

$$\tilde{\varepsilon}_r = (\frac{\omega\varepsilon}{\sigma})^2 = (\frac{\omega\varepsilon_0}{\sigma})^2 \varepsilon_r^2 \tag{29}$$

With this choice of perturbation in (29), Electric field wave equation is more simplified:

$$\tilde{\varepsilon}_r \tilde{E}'' + \tilde{E}' + \tilde{E} = 0 \tag{30}$$

As mentioned in section (II-E), this new perturbation parameter, $\tilde{\varepsilon}_r$ includes $(\omega,\ \sigma)$ parameters. Thus any "slight" or relative change on these parameters can also be treated as perturbation. Also 1^{st} derivative initial condition for unitless variable turns out to be;

$$\frac{\partial \tilde{E}'}{\partial \tau}\bigg|_{\tau=0} = \frac{1}{\tilde{\varepsilon}_r} \tag{31}$$

which is also unitless.

F. SOLUTIONS FOR THE UNITLESS EQUATION

In this section, unitless forms of the wave equation will be solved conventionally for a second order ODE, and then will be compared with the equation, solved without any unit conversion with the same method. Solutions will be compared and visualised on plots.
Unit transformation in (i) was shown to be fully consistent with the original solution. Below, this and an additional transformation (ii)will be shown to have the same result:

i) $t_0 = \frac{\sigma}{\omega^2\epsilon}$; $E_0 = \frac{\varepsilon_0 C}{\sigma}$ Transformation:
Has the same propagation attenuation and phase constants with the original equations. Analysis for these unitless parameters are given in (25) to (30). In Fig.3 solution plots for the original case and unitless versions are illustrated as to be the same, which validates the transformations for Electric field and time.

ii) $t_0 = \frac{1}{\omega}$; $E_0 = \frac{C}{\omega\varepsilon_r}$ Transformation:
In this transformation, method is the same, reference values that are used to non-dimensionalize the variables have the same units (which is [s] for t, [V/m] for E_β) but the combination of natural variables to obtain these units are different. Same strategy, same units but choosing reference values for t and E different, following wave equation solution obtained and shown to be completely matching with traditional solution of the initial equation. Reference values for t nd E_β are calculated and suitable values are found after some trial and error sessions. Results, without going into detail are given in 32. It should be noted that this process is applied to the wave equation which is space-transformed in Section 2.2.

Using the reference for t and E_β as follows, gives (30) :

$$t_0 = \frac{1}{\omega}, \quad \tau = \omega; \; E_0 = \frac{C}{\omega\varepsilon_r}, \quad \tilde{E} = \frac{\omega}{\varepsilon_r}C\tilde{E}$$

$$\begin{gathered}\varepsilon_r\tilde{E}'' + \frac{\sigma}{\omega\varepsilon_0}\tilde{E}' + \varepsilon_r\tilde{E} = 0 \\ \tilde{E}(0) = 0; \tilde{E}'(0) = 1\end{gathered} \tag{32}$$

In Fig. 3, unitless solution and reference solutions are plotted to be compared. They perfectly match on each other, proving usage of different references, for unitless solutions. With this conclusion, it is important to emphasize that making parameters unitless is not a unique process but has to be done properly in order to make overall equation and initial conditions simpler to be solved. Besides, this process enables to expand any complex equation into series, like Taylor or asymptotic series which are used in many different methods.

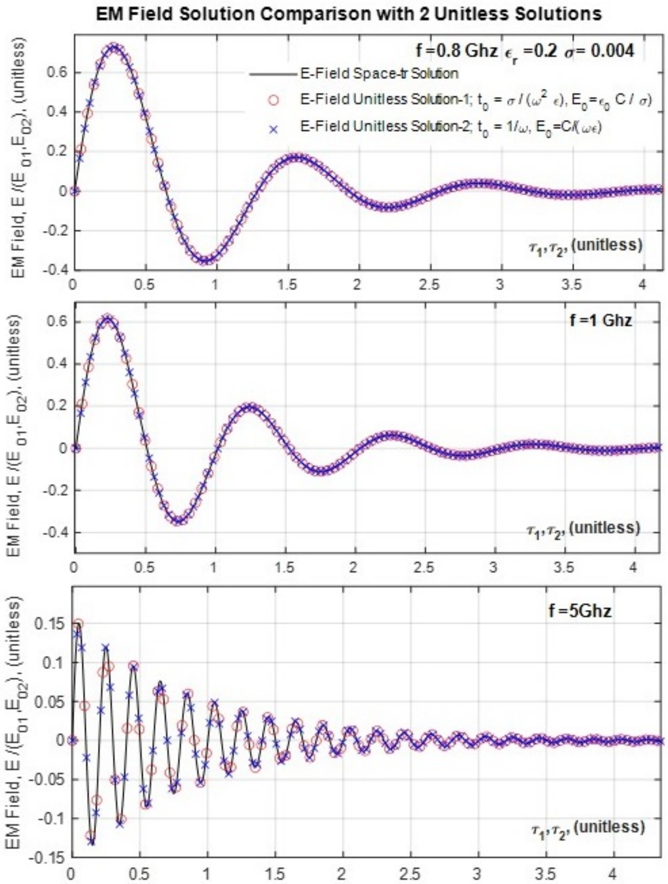

FIGURE 3. Verification for Unitless Lossy wave equation solution with different reference parameters

In the next section, method will be offered in order to solve the equation by perturbation series approach.

III. PERTURBATION METHOD SOLUTION

Our main goal was to find an approximate approach to the Electromagnetic differential equation to gain insight of the propagation behaviour inside ENZ material, especially for the small values of the permittivity. Due to the nonlinear nature of the problem, this analytic process can not be easily solved. One of the most important tool in approximating functions in some small neighbourhood is the Taylor's theorem. In general, Taylor theorem approaches the problem as;
Given a certain tolerance "$\varepsilon = |x - x_0| > 0$" how many terms should we include in the Taylor polynomial to achieve that accuracy?.
However, asymptotic approximation approaches the problem differently :
Given fixed number of terms N, how accurate is the asymptotic approximation as "$\varepsilon \to 0$?" [5]
This approach gives the perturbation method the advantage of converging the result much faster than Taylor series, with less number of terms [2]. This is the inherent nature

of asymptotic series, which has another big advantage to provide approximation even for divergent regions that can not be handled with Taylor series approach. As mentioned in Sect.1.3, perturbation theory deals with a set of different type of equations, in many different disciplines. [1], [4]. In this study, perturbation theory on a lossy electromagnetic wave is handled. As a starting point, general wave equation was given in (4),(5). A specific case of the wave equation, lossy electric field wave equation was considered (9).

First, Electric field (E_x) component is space transformed to (E_β) and then the units of the equation variables "V/m" for (E)" and "s" for (t) are cleared off by non-dimensionalization process. This operation requires new variables to be defined as ; $t \leftrightarrow \tau$ and $E \leftrightarrow E_0$:

Substitution of the variable equivalences with some algebraic operations, yields the simplified equation in (33):

$$\tilde{\varepsilon}_r \tilde{E}'' + \tilde{E}' + \tilde{E} = 0$$
$$\tilde{E}(0) = 0,\ \frac{\partial \tilde{E}'}{\partial \tau}\Big|_{\tau=0} = \frac{1}{\tilde{\varepsilon}_r} \tag{33}$$

where the unitless quantities (E_0, t_0 and the "super" perturbation parameter $\tilde{\epsilon_0}$ were found to be;

$$\tilde{\varepsilon}_r = (\frac{\omega\varepsilon}{\sigma})^2;\ t_0 = \frac{\sigma}{\omega^2\varepsilon};\ E_0 = \frac{\varepsilon_0 C}{\sigma} \tag{34}$$

Eqn.(33) is a quadratic Ordinary Differential Equation (ODE) that was solved conventionally in the first part. In order to show and analyse the effect of the perturbation parameter better, it will be solved by perturbation methods.

Before representing function in perturbation theory, a short description about asymptotic analysis will be given. Comprehensive information about Perturbation methods can be found in, [2], [17], [18], [19]. To have an idea about "what is happening with the function as the perturbation parameter approaches 0", asymptotic analysis has to be carried. In this work, perturbation parameter is the permittivity, ε_r and as a specific application, this permittivity has a value between [0-1], which is a subject of a popular concept, Epsilon Near Zero (ENZ). The usefulness of an asymptotic expansion arises from the fact that only a few terms of the series are required to give a good approximation to the function, whereas with a Taylor series expansion many terms are required for equivalent accuracy [20], [21]. Note that from the definition of an asymptotic expansion, the remainder after N terms is much smaller than the last term retained as $x \to x_0$. Generally, perturbation problems are grouped in two classes:

- Regular Perturbation Problems,
- Singular Perturbation Problems

Below, due to the scope of this work, singular type of perturbation problems will be detailed.

A. SINGULAR PERTURBATION PROBLEMS

In singular perturbation problems, there exists a small parameter (ε) which changes the order or degree of the problem so as to introduce a new class of solutions which is different from unperturbed problem. Thus, if the character of the problem changes discontinuously for $\varepsilon = 0$, in any region of the solution domain, than the problem is singular.

In this study, perturbation parameter impacts the term with the highest order; ($\tilde{\varepsilon}_r \tilde{E}'' + \tilde{E}' + \tilde{E} = 0$) and in unperturbed case i.e. when ($\tilde{\varepsilon} = 0$) highest order term disappears. This is an irregularity case and shall be treated differently. This is because the equation changes completely, decreases its order of derivation from "*2*" to "*1*". As a another definition, this irregularity is called as "Singularity". Singular behaviour forces the function to change rapidly, and has to be expanded in that region to analyse the big effect of the parameter in detail. In that sense, these rapid transition problems can be grouped as, Initial layer,boundary layer,Internal layer problems. In this study, the problem investigated is an initial layer type, in which the solution makes an immediate jump at the beginning, than, continue to smooth solution outside this "fast changing" region. That is the reason (E(0), E'(0)) will be used to solve the problem. The method used to solve such problems is basically to remove the singularity and to expand the interval where the perturbation parameter is dominant in asymptotic sense. That is a good coincidence to explain Dominant Balance Method for the study.

B. SOLUTION BY DOMINANT BALANCE METHOD

Dominant Balance Method; is a technique in perturbation theory, used to find a 'leading order' approximation $y_0(x;\varepsilon)$ for a singular equation, which asymptotically approaches the true solution $y(x)$ as $\varepsilon \to 0$ [14]. By Dominant Balance technique a scaling factor for the boundary value of the solution domain is introduced. This parameter "stretches" the variable among the boundary region, allowing to find a leading order term even for $\varepsilon \to 0$ This "stretching variable" can be defined as a function of ε as $\delta(\varepsilon)$ and in this work as perturbation parameter defined, $\tilde{\varepsilon} \to \delta(\tilde{\varepsilon})$. $\tilde{\varepsilon}$ represents "super perturbation parameter, defined in (29).

Since this technique solves the equation for a sufficiently short initial interval, which is inside the boundary layer, Electric field wrt. this scaled time variable called as "inner" and depicted as E_i. Rest of the solution function is called outer region, hence the result is E_0. A new time variable which is the stretched form of the original variable defined as "$\tilde{\tau}$" below in (35) as;

$\tilde{\tau} = \frac{\tau}{\delta(\tilde{\varepsilon})}$.

Using chain rule and for simplicity, δ instead of $\delta(\tilde{\varepsilon})$:

$$\tilde{E}' = \frac{1}{\delta}\frac{\hat{E}}{\tilde{\tau}} = \frac{1}{\delta}\hat{E}',\quad \tilde{E}'' = \frac{1}{\delta^2}\hat{E}''$$
$$\frac{\tilde{\varepsilon}}{\delta^2}\hat{E}'' + \frac{1}{\delta}\hat{E}' + \hat{E} = 0 \tag{35}$$

Taking the coefficients of the resultant equation, binary combinations will be asymptotically equalized and searched if they are much greater than (dominant to) the third component. Since there are three coefficients, 3 such pairs are possible to find. Below, the good working pair is presented only:

Coefficients of (35) : $\frac{\tilde{\varepsilon}}{\delta^2},\ \frac{1}{\delta},\ 1.$

Choosing first two as coefficient as dominant balance pair :

$$\frac{\tilde{\varepsilon}}{\delta^2} \sim \frac{1}{\delta}, \quad \delta = \tilde{\varepsilon}$$
$$\text{Thus, } \frac{1}{\tilde{\varepsilon}} \sim \frac{1}{\delta} \gg 1, \text{which is acceptable.} \tag{36}$$

Electric field with this new scaled variable is :

$$\hat{E}'' + \hat{E}' + \tilde{\varepsilon}\hat{E} = 0,$$
$$\text{with } \hat{E}(0) = 0; \ \hat{E}'(0) = 1 \tag{37}$$

Equation(37) is now in a regular format that can be solved with regular perturbation techniques. But the range of validity for this equation is limited to perturbation parameter scale, which is a very short interval. That is the reason, solution which is valid for this transform or stretched variable is called "inner" solution, that makes a sudden jump inside a very short distance. In fact this initial part of the function is very interesting and behaviour of the function in this region worth to be analysed and emphasized.

Rest of the solution is called "outer solution" for which there are a variety of techniques to solve this part, basically due to lack of initial conditions of the outer part. Below, 2 different techniques for outer solutions will be used. For both inner and outer solutions, asymptotic expansions upto first rank will be used in the calculations below:

i) Inner solution

Let inner solution of the Electric "field to be; $\hat{E}^i$ and outer to be $\hat{E}^o$. For the sake of simplicity, "ˆ " is dropped for inner and outer calculations.

$$E^i = E_0^i + \tilde{\varepsilon}E_1^i + O(\tilde{\varepsilon}^2) \tag{38}$$

Substituting (37) to (38) and grouping similar terms as coefficients of perturbation parameter, one obtains 2 sets of equations for $E_0^i \ and \ E_1^i$:

$$\frac{\partial^2 E_0^i}{\partial \tilde{\tau}^2} + \frac{\partial E_0^i}{\partial \tilde{\tau}} = 0,$$
$$E_0^i(0) = 0; \ E_0'^i(0) = 1 \tag{39}$$

and

$$\frac{\partial^2 E_1^i}{\partial \tilde{\tau}^2} + \frac{\partial E_1^i}{\partial \tilde{\tau}} = -E_0^i,$$
$$E_1^i(0) = 0; \ E_1'^i(0) = 0 \tag{40}$$

Solution to (39) is; $E_0^i = 1 - e^{-\tilde{\tau}}$. Putting this to inhomogenous differential equation (40) with initial conditions, gives the result as $E_1^i = 2 - \tilde{\tau} - (2+\tilde{\tau})e^{-\tilde{\tau}}$.

Combining $E_0^i \ and \ E_1^i$ into (41) gives the result for inner solution:

$$E^i(\tilde{\tau}, \tilde{\varepsilon}) = (1 - e^{-\tilde{\tau}}) + \tilde{\varepsilon}[(2-\tilde{\tau}) - (2+\tilde{\tau})e^{-\tilde{\tau}}] \tag{41}$$

ii) Outer solution

Main challenge for the outer solution that it does not satisfy boundary conditions, since it starts away from the initial point. Also due to perturbation parameter multiplying highest order term, its order is dropped for the first term (or, leading order term). Good news is, this reduced equation still valid due to slow variations and provides a good approximation to the original solution, as long as the inner and outer solutions are matched. This matching process is done in a variety of ways. Once again using asymptotic series expansion for outer part:

$$E^o = E_0^o + \tilde{\varepsilon}E_1^o + O(\tilde{\varepsilon}^2) \tag{42}$$

and organizing the equation similar to inner solution, following equation set for zeroth order ($\varepsilon^0 \to leadingterm; E_0^o$) and 1st order ($\varepsilon^1 \to; \ E_1^o$)term obtained:

$$\frac{\partial E_0^o}{\partial \tau} + E_0^o = 0 \tag{43}$$

General solution to leading order (43) is $E_0^o = Ae^{-\tau}$. For 1^{st} order term; following equation can be written:

$$\frac{\partial E_1^o}{\partial \tau} + E_1^o = -\frac{\partial^2 E_0^o}{\partial \tau^2} \tag{44}$$

Substituting E_0^o from (43) yields solution for this non homogeneous differential equation; $(-A\tau + B)e^{-\tau}$.

Summing up leading order and 1^{st} order terms, according to (42) gives the following result for outer solution:

$$E^o(\tau, \tilde{\varepsilon}) = Ae^{-\tau} + \tilde{\varepsilon}(-A\tau + B)e^{-\tau} \tag{45}$$

For outer solution, time variable is not converted to $\tilde{\tau}$. Since there is no valid initial condition for this equation it will be solved parametrically and these unknowns will be calculated by matching inner and outer solutions at the interception region. In this study, unknown "A" and "B" are found parametrically by using "patching" and "Van Dyke" methods. Both of them have their advantages and disadvantages and details of these solutions can be found comprehensively in the literature. [2], [12], [9].

By equating inner solution (39) and outer solution (43) at the boundary layer, and equating time scale which is done for the inner solution-dominant balance $\tilde{\tau} = \frac{\tau}{\tilde{\varepsilon}}$; a solution for the unknown parameters and resultant outer solution found as :

$$A = 1; \ B = 2; \text{ for } \tilde{\varepsilon} \ll \tau \ll \tilde{\varepsilon}^{1/2}$$
$$E^o(\tau, \tilde{\varepsilon}) = e^{-\tau} + \tilde{\varepsilon}(\tau + 2)e^{-\tau} \tag{46}$$

Perturbation solution is the superposition of the inner and outer solutions:

$$\hat{E}(\tau, \tilde{\varepsilon}) = \begin{cases} 1 - t - (1+t)e^{\frac{-\tau}{\tilde{\varepsilon}}} + 2\tilde{\varepsilon}(1 - e^{\frac{-\tau}{\tilde{\varepsilon}}}); 0 \le \tau \le \eta(\varepsilon) \\ e^{-t} + \tilde{\varepsilon}(-\tau + 2)e^{-\tau} \ for \ 1 \le t \le 3 \end{cases} \tag{47}$$

Solutions for this equation with different parameters are given in Fig. 5, Fig. 6, Fig. 7.

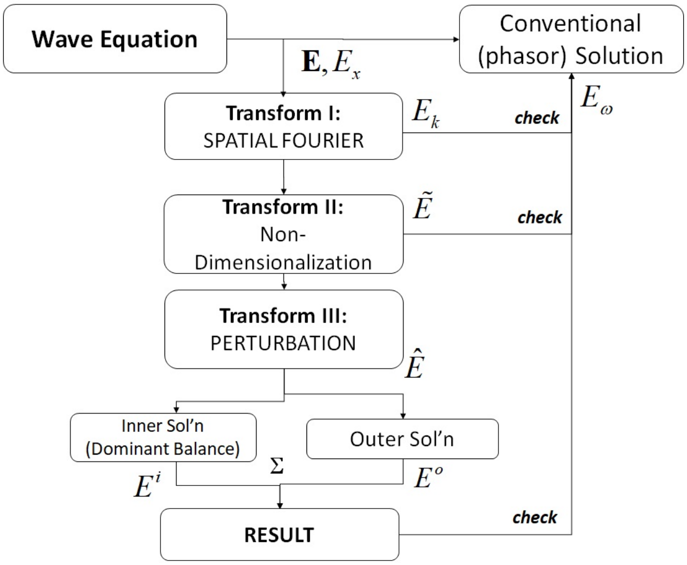

FIGURE 4. Flow diagram of the Study

IV. CONCLUSIONS

In this study, time domain response of an Epsilon Near Zero type material to a single frequency electromagnetic plane wave has been investigated by using perturbation methods. A general work flowdown and a list of transformation applied are given in Fig.4. Frequency domain analysis regarding the reflection characteristics of such materials at the vicinity of the resonance frequencies has been extensively analysed in the literature. On the other hand, time domain analysis regarding the behaviour of penetrating wave inside these materials is relatively quite rare. In this study, behaviour of the wave inside the material has been zoomed out by the powerful mathematical capabilities of Perturbation Analysis, which is a time domain method. Due to the nature of the problem and conductive behaviour of the material, plane wave fades out to zero in a very short time of order nanoseconds. Thus, detailed analysis have been carried for a very short time interval.
Three different intra-transformations that serve for the Perturbation solution result have been applied to a lossy plane wave equation:

- Spatial transform,
- Unitless transform,
- Dominant Balance (inner perturbation solution) transform.

For each transformation, transformed equation is solved with either conventional or novel methods, and compared with the original, conventional solution.

Effect of the parameters to the solution are investigated Fig. 5, Fig. 6, Fig. 7. In Fig. 5, ε_r increased in graphs (a), (b), (c) and seen that perturbation theory solution for the wave equation is a good approximation for small values of ε_r. This is good for the study, since the philosophy of Perturbation theory and Epsilon Near Zero Concept, both requires or "promotes" a relative permittivity which is close to zero. Thus decreasing the perturbation parameter towards zero, optimizes Perturbation solutions also.
There is another parameter that changes oscillation characteristics; Conductivity (σ). In Fig.6, conductivity constant, σ is changed while ε_r is kept constant. In (a), when the conductivity increased 10 times ($\sigma = 0.2$); it is observed that the results worse in the outer region. In (b) $\sigma = 0.1$ which is 5 times increased is also satisfactory. As the conductivity decreased, $\sigma = 0.004$) equation becomes oscillatory, which degrades the success of perturbation solution approximation In (c). Thus there is an optimum interval for the conductivity. Effect of the frequency is analysed in Fig.7. In these plots, operating frequency increased from 100 Mhz to 10 Ghz while $\varepsilon_r = 0.1$ and $\sigma = 0.02$ are kept constant. It is observed that when the frequency is low (100 Mhz), perturbation approach solution is not satisfactory (a). In (b), original case where the analysis carried are presented, and the result is quite satisfactory from perturbation approach point of view. In (c), frequency increased 10 times more and system starts to oscillate. This fact, as in the other cases, negatively affects perturbation solution which is clearly seen in (c).

As a general fact, the system oscillates if the propagation term

$$\tilde{\beta} = \frac{\sqrt{\left(\frac{\sigma}{\varepsilon_0}\right)^2 - 4{\varepsilon_r}^2\omega^2}}{2\varepsilon_r} \tag{48}$$

is imaginary, i.e. if it satisfies the following oscillation condition:

$$\varepsilon_r > \frac{\sigma}{2\varepsilon_0\omega} \tag{49}$$

Due to (48), (49) oscillation starts for increasing ε_r, frequency (ω), and decreasing σ. These facts are illustrated in Fig.5, Fig. 6, Fig. 7. In this work, oscillation effects negatively the accuracy of the perturbation theory due to the methods used so far. Epsilon Near Zero (ENZ) concepts depends on the permittivity behaviours of the of the materials depending on frequency. Mathematical models for dispersion of the materials are given by Lorentz Classical Theory (1878) and Drude Dispersion Model (1900). [4], [10]. In these models, especially around the plasma frequency of different materials, where the real part of the effective permittivity value of the material is approximately zero (50).

$$\omega_p = \sqrt{\frac{Ne^2}{m\epsilon_0}} \tag{50}$$

Increasing plasma frequency, Real[$\varepsilon(\omega)$] and Imaginary[$\varepsilon(\omega)$] increases, too. Thus, investigation of frequency dependence of the perturbation theory solution in Epsilon Near Zero Materials might constitute an interesting future work. Plasma frequency behaviour of different materials (Ag, Al, NiFe, etc) forces the solution to be carried on different frequencies. [3]. In this study, constant frequency wave penetration into a metamaterial has been investigated.As mentioned above, simultaneous injection of different frequency signals to a

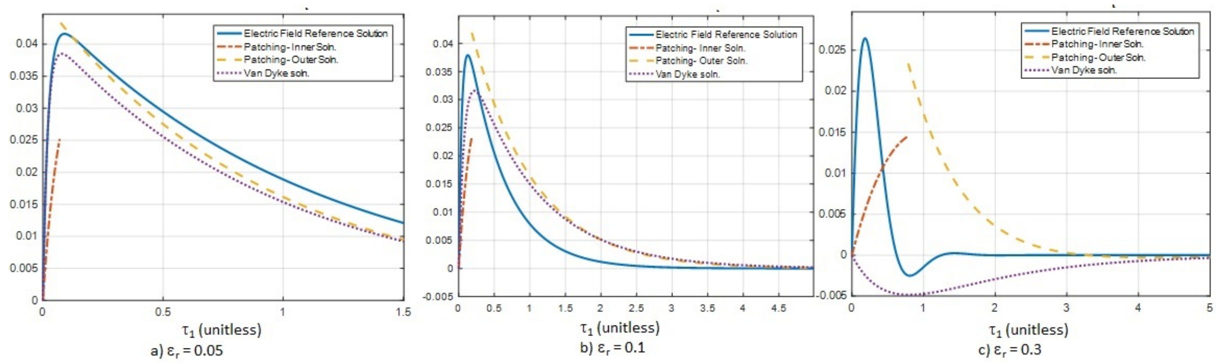

FIGURE 5. Effect of decrease in Perturbation parameter to the solutions when $\sigma = 0.02$, f=1 Ghz

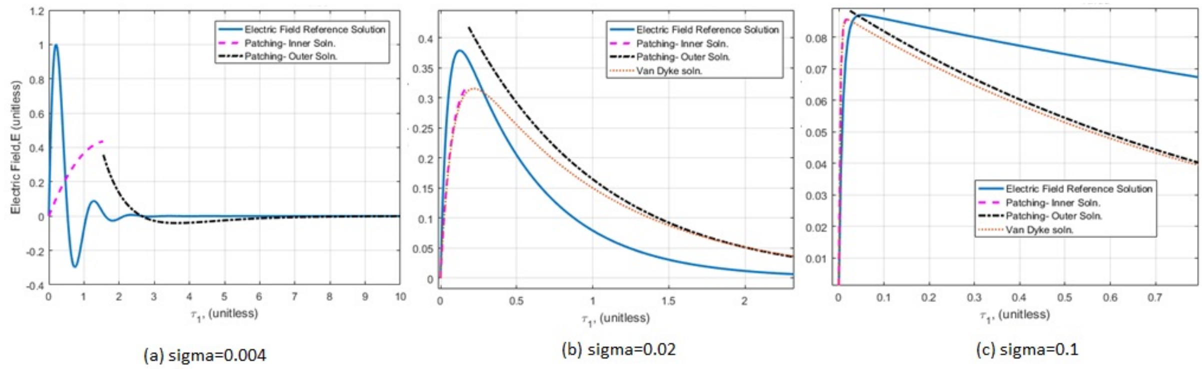

FIGURE 6. Effect of Conductivity parameter to the solutions when $\varepsilon = 0.1$, f=1 Ghz

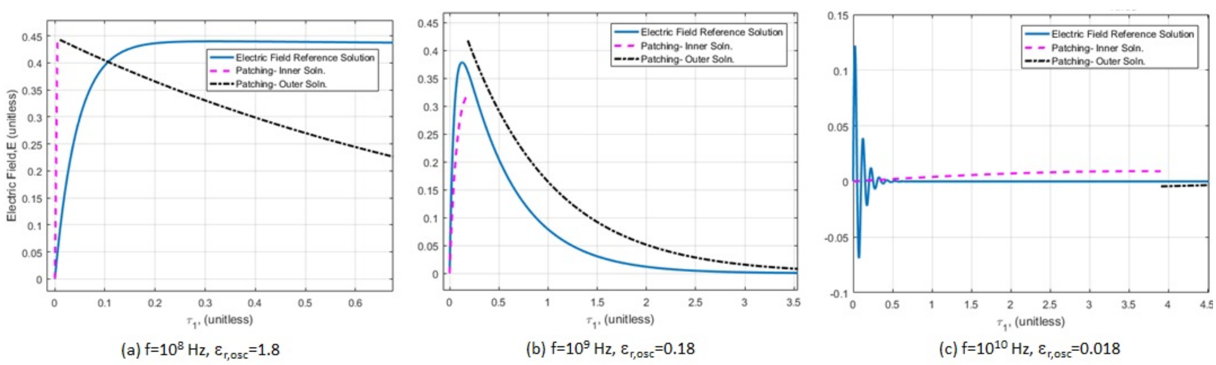

FIGURE 7. Effect of Frequency change to the solutions when $\varepsilon = 0.1$; $\sigma = 0.02$

metamaterial, interaction of these signal with each other to form harmonic signals in accordance with the dispersion characteristics, is left for a future work.

As another future work, more detailed solutions including higher order terms can be included to asymptotic series expansion and error behaviour may be investigated. Perturbation techniques to include periodic solution perturbations like Multi Scale can be applied for the oscillating cases. Other matching techniques like initial correction or WKB /(Wentzel-Kramers-Brillouin) techniques are also interesting topics to be studied. [21], [22]

V. ACKNOWLEDGMENT

Authors would like to express their gratitude to the anonymous referee, for his/her valuable comments and suggestions throughout the manuscript.

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